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Thouless quantum walks in topological flat bands

Carlo Danieli¹, Claudio Conti^{2,3}, Laura Pilozi^{1,3} and Valentina Brosco^{1,2}

Abstract

Non-Abelian gauge symmetries are cornerstones of modern theoretical physics, underlying fundamental interactions and the geometric structure of quantum mechanics. However, their potential to control quantum coherence, entanglement, and transport in engineered quantum systems remains to a large extent unexplored. In this work, we propose utilizing non-Abelian Thouless pumping to realize one-dimensional discrete-time quantum walks on topological lattices characterized by degenerate flat bands. Through carefully designed pumping cycles, we implement different classes of holonomic coin and shift operators. This framework allows for the construction of quantum walks that encode the topological and geometric properties of the underlying system. Remarkably, the resulting evolution exhibits parity symmetry breaking and gives rise to a dynamical process governed by a Weyl-like equation, highlighting the deep connection between parity and time-reversal symmetry breaking in quantum walks.

Introduction

Quantum walks (QWs) describe the quantum evolution of a system on a graph^{1–4}. As a fundamental concept in quantum information science and quantum kinetics, QWs have applications in network exploration, quantum information processing, and even biochemical systems^{5–7}. They provide a natural framework for studying quantum transport and diffusion, integrating concepts from graph theory, Markov processes, and quantum mechanics⁸. Originally introduced as the quantum counterparts of classical random walks, QWs exploit quantum interference and entanglement to shape the final statistical distribution and extend the walker's path length. Their computational universality^{2,9} makes them instrumental in the development of quantum algorithms^{3,8,10–13}, with applications in quantum search^{14,15}, simulation¹⁶, and quantum communication¹⁷.

QWs take place on graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the vertices $\mathcal{V}$ represent the sites the walker can visit, and the edges $\mathcal{E}$ define the connections between the sites. While space in QWs is discrete³, time can be either discrete or continuous—yielding continuous-time^{2,10} and discrete-

time¹⁸ QWs. The state of the walker is described by its position on the graph and a quantum coin variable, such as spin or photon polarization. In Discrete-Time Quantum Walks (DTQWs), the walker evolves through a sequence of unitary maps, each consisting of a coin operator $\mathcal{R}$, which acts on the coin state, and a conditional shift operator $\mathcal{T}$, which moves the walker depending on the coin state. The Hadamard is a common coin operator⁴, however, DTQWs can be defined with different parametric families of quantum coin operators for better control and optimization^{19,20}.

By appropriate choice of the coin operation and underlying graph topology, one can tailor the walk dynamics to implement problem-specific symmetries, introduce controlled disorder, or explore symmetry-breaking phenomena. Suitably designed DTQWs can simulate arbitrary quantum circuits, thereby achieving computational universality⁹. This universality is conditional in the sense that it depends on the specific configuration of the walk. The underlying graph must be appropriately engineered and the set of available coin operators must be sufficiently rich to generate a universal gate set. Under these constraints, the DTQWs formalism provides a complete and physically implementable model of quantum computation, equivalent in power to the standard circuit model.

In this work, we propose a general approach to construct DTQWs using non-Abelian holonomies. Holonomic

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transformations enable full control of the state of quantum systems and allow for the generation of topologically quantized transport on lattices²¹. We show that these two ingredients can be combined to engineer quantum walks with tunable features and investigate their inherent relation with non-Abelian Thouless pumping^{22–24}.

Thouless pumping is a paradigmatic topological phenomenon, yielding quantized transport in slowly and cyclically modulated one-dimensional lattices²¹. In the presence of degenerate Bloch bands, Thouless pumping acquires a non-Abelian character²². In this case, the system is initially prepared in a Wannier state belonging to a degenerate band and undergoes a geometric evolution dictated by the Wilczek-Zee connection²⁵.

Here, we employ non-Abelian Thouless pumping to generate discrete-time quantum walks. To keep the discussion simple, we illustrate our results on a Lieb chain with two degenerate flat bands^{26–29}. However, our approach can be extended in multiple directions—by considering alternative internal degrees of freedom for encoding the coin state, or by implementing lattices with different topologies and dimensionalities.

We show that holonomic transformations allow the realization of arbitrary unitary coin operations—going beyond the standard Hadamard coin—and unidirectional, conditional shift operators. We refer to this new class of one-dimensional quantum walks as *Thouless holonomic Quantum Walks* (ThQWs). The structure of ThQWs is intrinsically linked to the topological and geometric properties of the Hilbert space and can be engineered by suitably designing the elementary pumping cycles.

We show that ThQWs enable the selective breaking of parity or time-reversal symmetry, allowing the engineering of distinct final quantum-correlated states. The parity-breaking nature of ThQWs is further highlighted by relating the dynamical equation governing the walk to the Weyl equation. We focus on a possible implementation in photonic waveguide arrays^{22,30}, where the propagation coordinate z plays the role of time. However, our results are general and can be extended to other platforms, including cold atoms in optical lattices³¹ and superconducting nanocircuits³².

Results

Discrete time quantum walks are discrete processes that take place on a graph $\mathcal{G}$ consisting of vertices $\mathcal{V}$ and edges $\mathcal{E}$ connecting them. Each vertex is associated with a set of quantum coin variables σ . The wavefunction at time $t \in \mathbb{Z}$ is written as $|\Psi(t)\rangle = \sum_{\mathbf{n}} \sum_{\sigma} \psi_{\mathbf{n}}^{\sigma}(t) |\mathbf{n}, \sigma\rangle$, where $\mathbf{n}$ labels the vertex $v \in \mathcal{V}$. A single time-step, advancing from t to $t + 1$ consists of two unitary operations: a coin operator $\mathcal{R}$ that couples the coin states within each vertex of the graph, and a conditional shift operator $\mathcal{T}$ that moves the walker along the edges conditionally to its coin level σ .

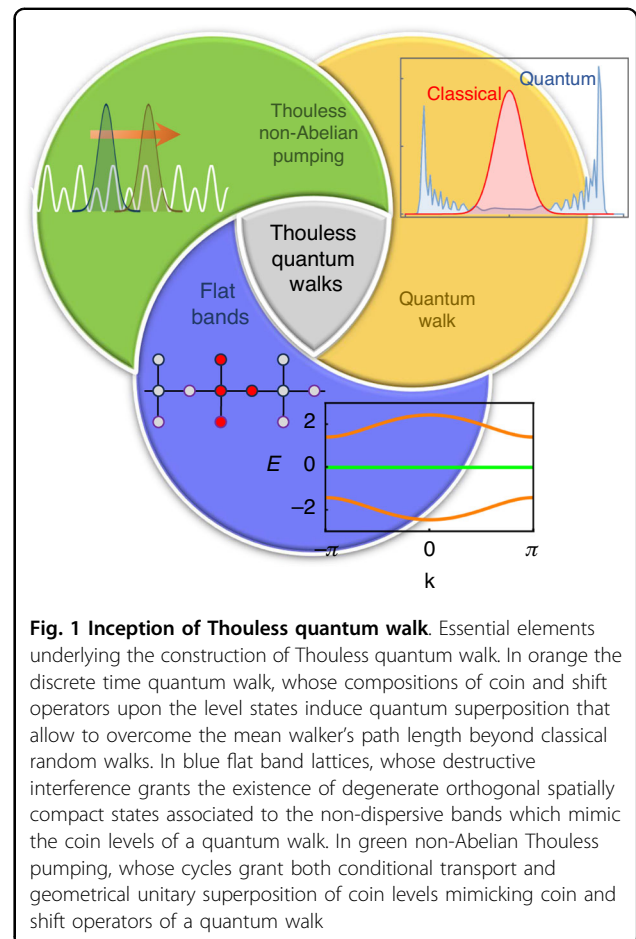


Fig. 1 Inception of Thouless quantum walk. Essential elements underlying the construction of Thouless quantum walk. In orange the discrete time quantum walk, whose compositions of coin and shift operators upon the level states induce quantum superposition that allow to overcome the mean walker's path length beyond classical random walks. In blue flat band lattices, whose destructive interference grants the existence of degenerate orthogonal spatially compact states associated to the non-dispersive bands which mimic the coin levels of a quantum walk. In green non-Abelian Thouless pumping, whose cycles grant both conditional transport and geometrical unitary superposition of coin levels mimicking coin and shift operators of a quantum walk

Although DTQWs have been realized in various platforms^{33–38}, their experimental implementation still poses significant technical challenges related to preserving discreteness in both time and space.

In this work, we introduce ThQWs as a robust framework to overcome these limitations, providing a scalable and topologically protected approach to realizing DTQWs in physical systems. Thouless quantum walks, or ThQWs, emerge from the interplay of two key ingredients: *degenerate flat bands* and *non-Abelian Thouless pumping*²², as illustrated in Fig. 1.

Flat-band lattices are translationally invariant tight-binding networks in which one or more Bloch bands are dispersionless—i.e., their energy remains constant and independent of the wavevector $\mathbf{k}$ ^{26–29}. These flat bands arise from destructive interference, giving rise to eigenstates that are strictly confined in space, known as compactly localized states (CLS).

Thouless pumping, on the other hand, refers to the quantized transport of particles driven by the adiabatic and periodic modulation of a confining lattice potential^{21,24}. In systems with degenerate Bloch bands, this process gives rise to a non-Abelian gauge structure

associated with local rotations within the degenerate subspace^{22,23}.

As we show below, these two components allow for the implementation of the basic building-blocks of DTQWs. Specifically, the degenerate CLS replicate the coin levels while non-Abelian pumping enables both quantized conditional transport and rotations in the coin space, analogous to the shift $\mathcal{T}$ and coin $\mathcal{R}$ operators. Interplay of flat bands and adiabatic transport is particularly relevant in chiral flat bands³⁹, where bipartite symmetry ensures the presence of multiple strictly flat bands at $\kappa_0=0$ independent of the hopping strengths. In these systems, the hopping modulation does not break the degeneracy, which is essential for the non-Abelian pumping.

Let us discuss this mechanism in details upon the most commonly studied DTQWs, where the walker moves along a line with site index n and has two coin levels $\sigma = \pm$ ¹⁸. In this case, the shift and coin operators are:

$$\mathcal{T} = \sum_n [|n + \delta_+\rangle\langle n| \otimes |+\rangle\langle +| + |-\rangle\langle -| \otimes |n + \delta_-\rangle\langle n|] \quad (1)$$

$$\mathcal{R} = \sum_n |n\rangle\langle n| \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (2)$$

where the integers δ_σ denote the shifts of the states $|n, \sigma\rangle = |n\rangle \otimes |\sigma\rangle$ while θ describes the superposition between the two coin levels.

Implementing $\mathcal{T}$ and $\mathcal{R}$ by means of non-Abelian Thouless pumping requires: (i) considering a lattice with $d_v = 2$ degenerate flat bands and choosing two orthogonal compact states, $|p_n\rangle$ and $|q_n\rangle$ localized within a unit cell n , that play the role of coin states; and (ii) designing two pumping cycles $\mathcal{C}_\mathcal{T}$ and $\mathcal{C}_\mathcal{R}$ of period $\lambda_\mathcal{T}$ and $\lambda_\mathcal{R}$ respectively, which yield conditional shift and coin operators. The composition $\mathcal{C} = \mathcal{C}_\mathcal{T} \circ \mathcal{C}_\mathcal{R}$ yields one cycle of duration $\lambda = \lambda_\mathcal{T} + \lambda_\mathcal{R}$.

The discrete nature of the resulting ThQWs is enforced by the topology of the lattice and the geometric structure of the pumping cycle. This process naturally generates a quantized motion made of steps connecting one unit cell to the next, each step taking one pumping period λ . Hence, the pumping period λ represents the ThQWs time unit. The state of the walker after t steps with $t \in \mathbb{Z}$ is described by the wavefunction $|\Psi(z_t)\rangle = \sum_n [\psi_n^p(z_t)|p_n\rangle + \psi_n^q(z_t)|q_n\rangle]$ with $z_t = z_0 + \lambda t$, and $|\Psi(z_{t+1})\rangle = \mathcal{C}|\Psi(z_t)\rangle$.

The geometric properties of ThQWs manifest in the displacement matrix D , which—as discussed in the Methods section and demonstrated in ref.²²—can be linked to the non-Abelian field strength. This matrix D characterizes the displacement undergone by an arbitrary superposition of the two coin states. Importantly, the

trace of the displacement matrix D yields the first Chern number of the degenerate band, $C_1 = \text{Tr}[D]$, which governs the displacement of the geometric center of the final distribution via the relation $n_t = \frac{C_1 t}{2}$.

We consider the implementation of ThQWs in a chain with two degenerate flat bands²², reminiscent of the two-dimensional Lieb lattice^{26–29}. The Hamiltonian features four sites per unit cell indicated respectively as a , b , c and d in Fig. 2a and it reads

$$H = \sum_n (J_{b1}a_n^\dagger b_n + J_{b2}a_n^\dagger b_{n-1} + J_c a_n^\dagger c_n + J_d a_n^\dagger d_n + \text{H.c.}) \quad (3)$$

where $x_n^\dagger$ and x_n denote the creation and the annihilation operators on sites $x = a, b, c, d$ of the cell n . For any choice of the hopping parameters, the spectrum consists of two non-dispersive modes with longitudinal momentum $\kappa_0 = 0$ and two dispersive modes with longitudinal momenta $\kappa_\pm(k) = \pm \Delta(k)$ with $\Delta(k) = \sqrt{J_c^2 + J_d^2 + |J_b(k)|^2}$ and $J_b = J_{b1} + J_{b2}e^{ik}$. The chiral symmetry of the lattice³⁹ implies that the Bloch states corresponding to κ_0 can be written as

$$\begin{aligned} |\phi_1\rangle &= \frac{J_c|d_k\rangle - J_d|c_k\rangle}{\rho} \\ |\phi_2\rangle &= \frac{-\rho^2|b_k\rangle + J_b^*(J_c|c_k\rangle + J_d|d_k\rangle)}{\rho\Delta(k)} \end{aligned} \quad (4)$$

with $\rho = \sqrt{J_c^2 + J_d^2}$. Let us assume that the system is initially prepared in a Wannier state $|\Psi(z_0)\rangle = \sum_{k,\ell} \alpha_\ell |\phi_\ell(k)\rangle e^{ikn}$. An adiabatic pumping cycle $\mathcal{C}$ acting on $|\Psi(z_0)\rangle$ then yields the state

$$|\Psi(z_1)\rangle = \sum_{k,\ell m} \alpha_\ell [W_\mathcal{C}(z_1, z_0)]_{m\ell} |\phi_m(k, z_0)\rangle e^{ikn} \quad (5)$$

where $W_\mathcal{C}(z_1, z_0) = \mathcal{P} \exp \left[i \int_{z_0}^{z_1} \Gamma_z dz \right]$ indicates the holonomy transformation associated with the Wilczek-Zee connection $[\Gamma_z]_{\ell m} = \langle \phi_\ell(k, z) | i \partial_z | \phi_m(k, z) \rangle$. Here $\mathcal{P}$ denotes the path ordering, while the indexes $m, \ell = 1, 2$ enumerate the basis states of the degenerate subspace and k runs over the reciprocal space vectors – see the Methods section for details.

To engineer the ThQWs we initialize the lattice setting $J_c = J_d = J_{b1} = 0$ and $J_{b2} = \mathcal{J}$. The Wannier states belonging to the non-dispersive modes in Eq. (4) can be then cast as

$$|q_n\rangle = \frac{|c_n\rangle - |d_n\rangle}{\sqrt{2}} \quad |p_n\rangle = \frac{|c_n\rangle + |d_n\rangle}{\sqrt{2}} \quad (6)$$

shown in Fig. 2b. These states act as the two coin levels. Following ref.²², and as discussed in more detail in the Supplementary material, the pumping cycles implementing conditional shift and coin operators on these states can be straightforwardly designed. As shown in Fig. 2c,



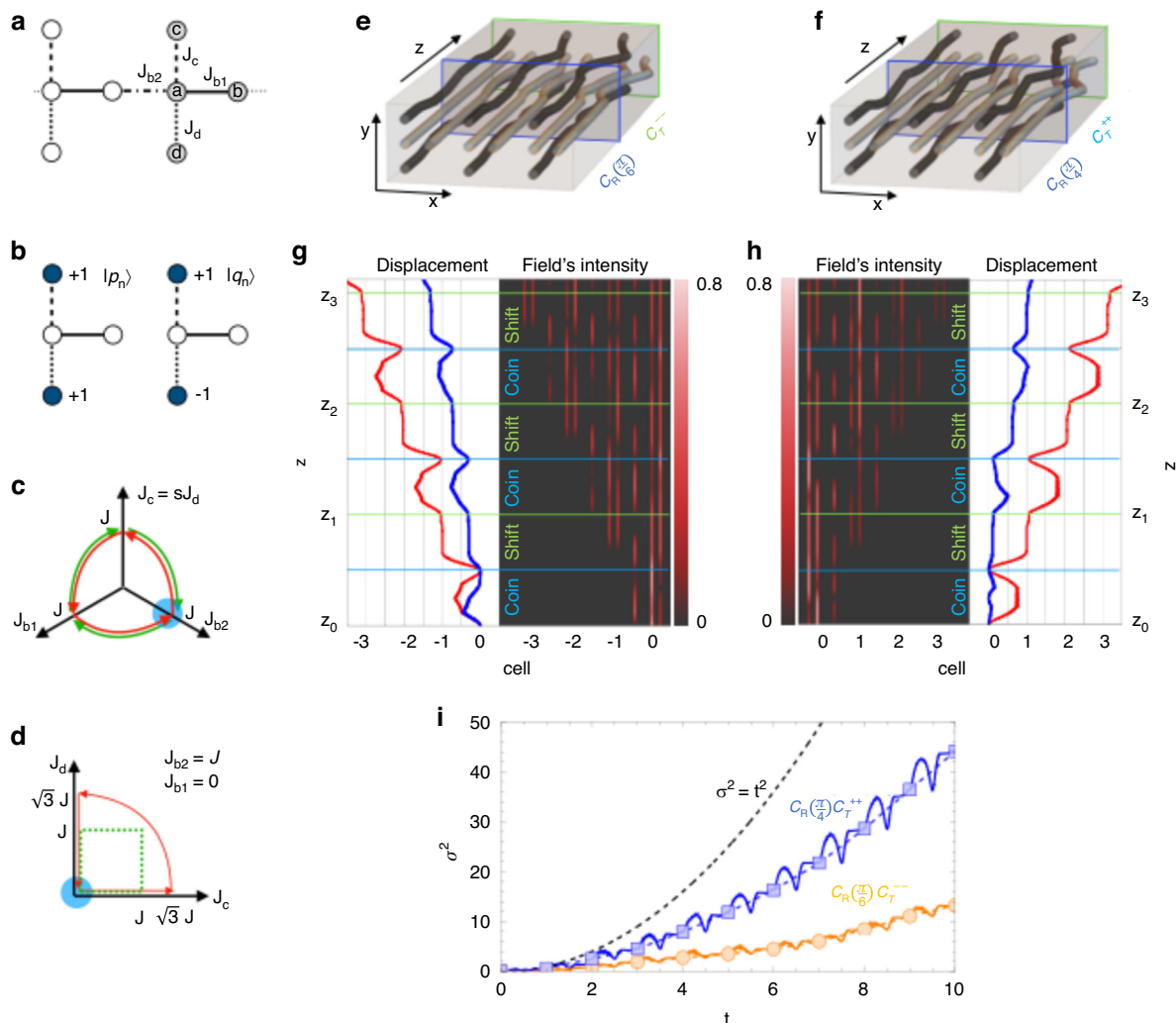


Fig. 2 Photonic implementation of a sample uni-directional Thouless quantum walk. **a** Flat band lattice profile with the unit cell colored in gray. **b** Orthogonal symmetric $|p_n\rangle$ and antisymmetric $|q_n\rangle$ states with non-zero amplitudes colored in dark blue. **c** Pumping cycles $\mathcal{C}_T^{+s}$ (red) and $\mathcal{C}_T^{-s}$ (green) in the parameter space. The blue circles indicate the initial point. **d** Same as (c) for pumping cycles $\mathcal{C}_R(\frac{\pi}{4})$ (red) and $\mathcal{C}_R(\frac{\pi}{6})$ (green). **e** Illustration of three unit cells of the pumped lattice implementing a time-unit formed by a coin $\mathcal{C}_R(\frac{\pi}{6})$ and a shift $\mathcal{C}_T^{++}$. **f** Same as (e) with coin $\mathcal{C}_R(\frac{\pi}{4})$ followed by a shift $\mathcal{C}_T^{++}$. **g** Three steps propagation of the ThQWs generated by a step $\mathcal{C}_T^{--}\mathcal{C}_R(\frac{\pi}{6})$ from a single-cell excitation $|\Psi(z_0)\rangle = \frac{1}{\sqrt{5}}|p_0\rangle + \frac{2}{\sqrt{5}}|q_0\rangle$. In both cycles $\mathcal{C}_T$ in (c) and $\mathcal{C}_R$ in (d) we set $\lambda\mathcal{J} = 200$. The green lines indicate the time-steps z_t , while the blue ones separate the coin for the shift. The right panel shows the trace of the displacement matrix D (red) and the center of mass (blue). **h** Same as (g) for a ThQWs generated by a step $\mathcal{C}_T^{++}\mathcal{C}_R(\frac{\pi}{4})$. **i** Variance σ^2 for $\mathcal{C}_T^{--}\mathcal{C}_R(\frac{\pi}{6})$ (orange) and $\mathcal{C}_T^{++}\mathcal{C}_R(\frac{\pi}{4})$ (blue) from $|\Psi(z_0)\rangle = \frac{1}{\sqrt{2}}|p_0\rangle + \frac{1}{\sqrt{2}}|q_0\rangle$. The squares and circles indicate the σ^2 at integer multiples of λ . The dashed black line guides the eye, indicating $\sigma^2 = t^2$

the conditional shift operator cycles, denoted as $\mathcal{C}_T^{s\xi}$, are represented by spherical triangles on the hyperplane $J_c = sJ_d$ with $s = \pm$, having anti-clockwise ($\xi = +1$) or clockwise ($\xi = -1$) orientation. In contrast, the coin operator cycle $\mathcal{C}_R$, shown in Fig. 2d, lies on the plane $\{J_{b2} = \mathcal{J}, J_{b1} = 0\}$. The corresponding holonomic transformations in k -space are

$$W_{\mathcal{C}_T^{s\xi}} = e^{i\xi\frac{\sigma_x}{2}(\sigma_0 - s\sigma_z)} \quad W_{\mathcal{C}_R} = e^{i\theta(\sin k\sigma_x + \cos k\sigma_y)} \quad (7)$$

with σ_i and σ_0 denoting the Pauli matrices and the identity. The above equations show that ThQWs have two important features: (i) they yield conditional shifts determined by the topology of the driven bands; and (ii) they offer the possibility to control the shift's direction by controlling the orientation of the pumping cycles, related to the index ξ . Moreover, in the specific example discussed here, they allow us to select which of the coin states, $|p_n\rangle$ or $|q_n\rangle$, moves by changing the relative phase

of J_c and J_d . Specifically, they implement the following operators:

$$C_T^{\xi+} : \sum_n [|p_{n+\xi}\rangle\langle p_n| + |q_n\rangle\langle q_n|] \quad (8)$$

$$C_T^{\xi-} : \sum_n [|p_n\rangle\langle p_n| + |q_{n+\xi}\rangle\langle q_n|] \quad (9)$$

which respectively result in right- and left- oriented uni-directional conditional shift operators. On the other hand, the cycle C_R yields zero net displacement while performing a rotation by an angle θ which is dependent on its precise shape²². The angle θ induces the superposition between the two flat bands. For example, the cycle in Fig. 2d corresponds to $\theta = \frac{\pi}{4}$ or $\theta = \pi/6$. In this case, reversing the orientation of the pumping cycles reverses the rotation. Composing the cycles $C_T^{\xi\pm}$ and C_R enables the realization of directional ThQWs with chirality $\chi = \xi$.

To demonstrate this, in Fig. 2e–h we show two examples of ThQWs corresponding, respectively, to $C = C_T^- C_R$ with $\theta = \pi/6$ (Fig. 2e, g) and $C = C_T^{++} C_R$ with $\theta = \pi/4$ (Fig. 2f, h). In Fig. 2e, f we show the schematic of three unit cells of the pumped lattice Eq. (3) where the blue lines mark the change between coin C_R and shift $C_T^{\xi\pm}$ and the green line mark the time-steps z_t . In Fig. 2g, h we display the field intensity along the waveguides for a single unit cell initial condition of the form $|\Psi(z_0)\rangle = \frac{1}{\sqrt{5}}|p_0\rangle + \frac{2}{\sqrt{5}}|q_0\rangle$. In our simulations, we set $\lambda\mathcal{T} = 200$, compatible with the parameters choices done in photonics implementations^{40,41} – see Methods for details. As one can see, ThQWs enable a remarkable control on field's propagation. The left and right panels of Fig. 2g, h display the displacement of the center of mass of the distribution (blue curve) and the trace of the displacement matrix D (red curve). As expected, only the latter is perfectly quantized, as it is directly related to the first Chern number – see Methods for details.

The variance σ^2 of a discrete time quantum walk distribution is a key indicator of its spreading behavior compared to a classical random walk. Crucially, while the variance of a classical random walk grows linearly $\sigma^2 \sim t$ with the number of steps t , a quantum walk exhibits quadratic growth⁴², $\sigma^2 \sim t^2$. Fig. 2i illustrates this ballistic spreading, showing the variance σ^2 for two different ThQWs: one defined by the time-step $C_T^- C_R(\frac{\pi}{6})$ (orange) and another by $C_T^{++} C_R(\frac{\pi}{4})$ (blue), analogous to Fig. 2e, f. Both ThQWs start from a symmetric single-cell excitation $|\Psi(z_0)\rangle = \frac{1}{\sqrt{2}}|p_0\rangle + \frac{1}{\sqrt{2}}|q_0\rangle$ and evolve for ten time-steps. The solid curves indicate the variance σ^2 during the evolution, while squares and circles indicate the value of σ^2 at the multiples of λ . As shown, the coin parameter θ significantly influences the walk's dynamics by modulating the effective spreading rate⁴². Indeed, the variance follows $\sigma^2 \simeq ft^2$ where the prefactor f depends on the coin

angle θ as well as on the initial state. By tuning θ , one can tailor the quantum walk for the desired variance—a property that can be exploited to enhance the efficiency of quantum search and sampling algorithms. In ThQWs, this tuning can be done by properly designing the pumping cycle C_R that implements the coin operator.

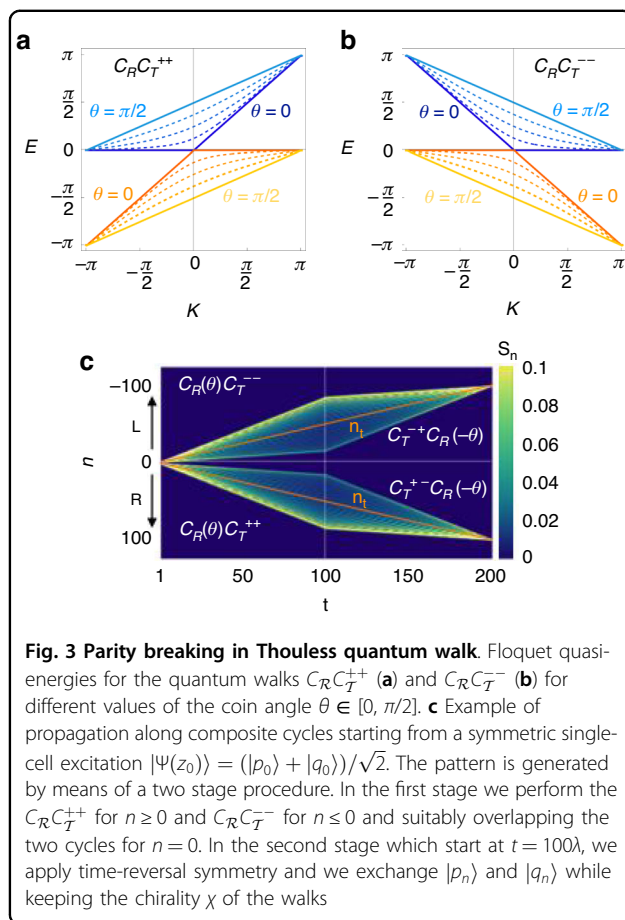
To illustrate the parity-broken nature of ThQWs, we analyze the dynamic structure of the quantum walk equations in the continuous time limit, as described in ref.⁴³. By doing so, for the quantum walk generated by the cycles $C_T^{\xi\pm} C_R(\theta)$, we obtain

$$\partial_t \Psi = -\xi \left[\cos \theta \frac{\sigma_0 + s\sigma_z}{2} + \sin \theta \frac{i\sigma_y + s\sigma_x}{2} \right] \partial_n \Psi + [(\cos \theta - 1)\mathbb{I} + i \sin \theta \sigma_y] \Psi \quad (10)$$

with $\Psi = (\psi_n^p(t), \psi_n^q(t))^T$ – see Supplementary material for details. For $\xi = \pm 1$ the above equation accounts for a right-moving(+) or left-moving(-) ThQWs. In the limit $\theta = 0$, for each value of ξ Eq. (10) describes a right-handed ($\chi = +1$) and a left-handed ($\chi = -1$) Weyl particle. Equivalently, the two Floquet quasi-energies E^{44} , see Supplementary material, associated with the processes $C_R C_T^{++}$ and $C_R C_T^{--}$ produce two anisotropic Floquet bands whose structure depends on the angle θ , as shown in Fig. 3a, b. For $\theta = 0$, the system exhibits a flat Floquet band $E = 0$ and a linearly dispersive Floquet band $E = \xi k$ which are respectively associated with $|q_n\rangle$ and $|p_n\rangle$ in the cycle $C_R C_T^{++}$ and the reverse for the cycle $C_R C_T^{--}$. For $\theta \neq 0$, the states $|p_n\rangle$ and $|q_n\rangle$ become coupled, yet the left-right directionality is preserved. Linear Floquet band dispersion $E = \frac{\xi}{2}(k \pm \pi)$ is restored for $\theta = \pi/2$ and the two states move in unison with half the group velocity of a $\theta = 0$ single state.

By composing ThQWs with different parities, more complex patterns can be generated. For instance, the pattern in Fig. 3c is produced through a two-stage procedure in which the ThQWs are combined with their counterparts obtained via the time-reversal operator and a chirality-preserving spin-inversion operator. In the first stage, the cycles $C_R C_T^{++}$ (for $n \geq 0$) and $C_R C_T^{--}$ (for $n \leq 0$) are performed, with an appropriate overlap at $n = 0$ to ensure smooth connectivity. In the second stage, which begins after 100 steps, time-reversal symmetry is applied, and the roles of $|p_n\rangle$ and $|q_n\rangle$ are exchanged while preserving the chirality χ of the time-reversed walks. This therefore involves combining ThQWs generated by the cycles $C_R(\theta) C_T^{++}$ and $C_R(\theta) C_T^{--}$ with those generated by $C_T^+ C_R(-\theta)$ and $C_T^- C_R(-\theta)$, respectively. The whole process maps the initial state $(|p_0\rangle + |q_0\rangle)/\sqrt{2}$ onto the state $(|q_{-t/2}\rangle + |p_{t/2}\rangle)/\sqrt{2}$ therefore generating quantum correlations between spin and position coordinates. Also in this case, the displacement of the middle point of the two distributions is governed by topology.





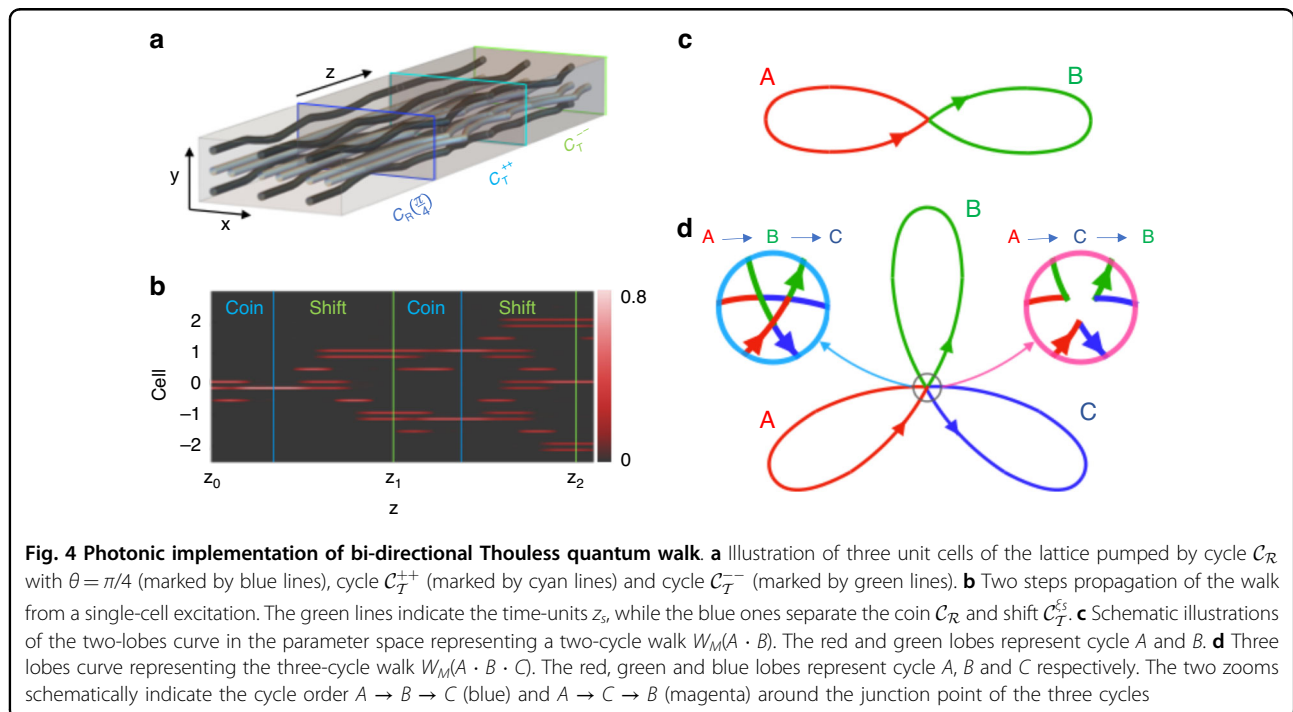
The high degree of control offered by ThQWs enables the realization of different classes of quantum walks. For example, varying θ periodically in the cycle C_R at every step z_t yields Floquet quantum walks⁴⁵, while varying θ with n spatially modulates ThQWs, yielding topologically nontrivial states – e.g. by designing a split-step advancement operator $C \equiv C_T^- C_R(\theta(n)) C_T^+ C_R(\theta(n))$ as discussed in ref.⁴⁶.

The pumping cycles $C_T^{\xi s}$ in Eq. (7) can be used to generate ThQWs with restored parity symmetry. For instance, composing cycles $C_T^{\xi+}$ and $C_T^{\xi-}$ for $\bar{\xi} = -\xi$ yields the shift operators

$$C_T^{\xi} = C_T^{\xi+} \circ C_T^{\xi-} : \sum_n [|p_{n+\xi}\rangle \langle p_n| + |q_{n-\xi}\rangle \langle q_n|] \quad (11)$$

In Fig. 4a we show three unit cells of pumped lattice waveguides, where this shift is composed with a coin C_R with $\theta = \pi/4$. Fig. 4b then shows the field intensity along the waveguides resulting from a single-cell excitation with $|\Psi(z_0)\rangle = \frac{1}{\sqrt{5}}|p_0\rangle + \frac{2}{\sqrt{5}}|q_0\rangle$. The transitions from the coin operator C_R to the shift operator C_T are marked by the blue lines, while the time-units z_1 and z_2 are marked by the green lines.

The freedom to combine and engineer different pumping cycles offers the possibility to probe the underlying non-Abelian gauge structure. Due to the non-Abelian multiband nature of the adiabatic evolution, the final probability distribution generated by the quantum walk depends on the order in which the shift and rotation cycles – denoted as C_T and C_R – are applied. In the asymptotic limit of many steps,



swapping the order of these cycles effectively corresponds to initiating the walk from a different quantum state, which can significantly alter the resulting distribution.

More striking signatures of the geometric and topological character of ThQWs emerge when unit steps are composed of at least three non-commuting pumping cycles. In such cases, only cyclic permutations of the cycle sequence yield quantum walks that are asymptotically equivalent – up to a change in the initial state—while other permutations result in fundamentally distinct quantum walks. This behavior is illustrated in Fig. 4c, d. For a generic two-cycle quantum walk $W(A \cdot B)$, the trajectory traces M times a two-lobe curve in the parameter space shown in Fig. 4c whose topology remains invariant under permutation of A and B . In contrast, for a three-cycle quantum walk $W_M(A \cdot B \cdot C)$, the trajectory forms a three-lobe curve in the parameter space. Changing the cycle order leads to inequivalent walks, each generating qualitatively different dynamical behavior—see Supplementary material for more details. In Fig. 4d the two possible orders are shown in the two zooms around the intersection point – namely, $A \rightarrow B \rightarrow C$ (blue left zoom) and $A \rightarrow C \rightarrow B$ (magenta right zoom).

Considering one-dimensional lattices with $d_v \geq 3$ degenerate flat bands may allow for the implementation of ThQWs with three or more coin levels. One way could be extend the tripod uni-cell shown in Fig. 2a to a M -pod unit cell and build a lattice that respects the bipartite symmetry³⁹. This symmetry relation guarantees $M - 2$ flat bands at $\kappa_0 = 0$ energy. This implementation of ThQWs through chiral flat band lattices is not restricted to DTQWs on a line, but could be exploited to design ThQWs on general graphs.

Discussion

Non-Abelian holonomies and geometric phases have emerged as powerful tools in theoretical physics, particularly within gauge theories, topological phases of matter, and quantum computation. In this work, we show that non-Abelian Thouless pumping provides a framework for realizing quantum walks with coin and shift operators defined through non-Abelian holonomies. Specifically, we demonstrate that in a lattice with multiple degenerate flat bands, Thouless pumping enables the implementation of holonomic quantum gates that act on the spatial degrees of freedom of a quantum particle and entangle them with its internal coin states. The topological origin of the pumping ensures quantized transport and can be naturally related to a class of discrete time quantum walks that we called Thouless Quantum Walks, or ThQWs.

ThQWs allow for a high-degree of control and tunability. Specifically, we show that they can be engineered to selectively break parity or time-reversal symmetry yielding a directionally chiral evolution that is effectively described

by Weyl-like equations in the continuum limit—mirroring relativistic dynamics but in a lattice-based, topologically protected framework. The topological and geometric nature of Thouless pumping endows ThQWs with robustness against certain kinds of noise and dynamical perturbations, a highly desirable property for quantum technologies see Methods. Moreover, we reveal that the combination of parity and time-reversal symmetry breaking allows for the emergence of well-defined quantum correlations in the long-time limit. This feature can be used to control entanglement patterns and engineer specific correlation structures, potentially useful for quantum communication protocols or quantum simulations.

Our approach not only opens a novel direction for the application of non-Abelian holonomies in dynamic quantum systems but also lays the groundwork for extending these ideas beyond the one-dimensional case studied here. Future generalizations may include higher-dimensional quantum walks, which could underlie lattices with higher Chern numbers, or systems with larger internal (coin) spaces, thus enabling more complex forms of quantum information encoding and manipulation.

Methods

Geometric properties of ThQWs

The displacement generated in a quantum walk for the system initially prepared in an arbitrary superposition of two CLSs centered on site n belonging to the level κ_v is given by

$$\Delta x = \sum_{m\ell} \alpha_m^* \alpha_\ell D_{m\ell}^\nu \quad (12)$$

where the coefficients α_ℓ define the initial state

$$|\Psi(z_0)\rangle = \sum_{k,\ell} \alpha_\ell |\phi_\ell(k)\rangle e^{ikn} \quad (13)$$

This Wannier state is defined for the normalized Bloch states $|\phi_\ell(k)\rangle$. In this sum, the coefficients α_ℓ define the superposition of the Bloch states of the $E = 0$ degenerate subspace for the index $\ell = 1, 2$, while k runs over the reciprocal space. After an adiabatic pumping cycle C , the state $|\Psi(z_0)\rangle$ in Eq. (13) turns to

$$|\Psi(z_1)\rangle = \sum_{k\ell m} \alpha_\ell [W_C(z_1, z_0)]_{m\ell} |\phi_m(k, z_0)\rangle e^{ikn} \quad (14)$$

where $W_C(z_1, z_0) = \mathcal{P} \exp \left[i \int_{z_0}^{z_1} \Gamma_z^z dz \right]$ indicates the holonomy transformation associated with the Wilczek-Zee connection $[\Gamma_z]_{\ell m} = \langle \phi_\ell(k, z) | i \partial_z | \phi_m(k, z) \rangle$. Here $\mathcal{P}$ denotes the path ordering, while the indexes $m, \ell = 1, 2$ enumerate the degenerate basis states. The displacement



matrix D is a square matrix which can be written as

$$[D]_{m\ell} = \frac{1}{2\pi} \int_{z_0}^{z_1} dz \int_{-\pi}^{\pi} dk [W_C^\dagger \mathcal{F}_{kz} W_C]_{m\ell} \quad (15)$$

with $\mathcal{F}_{kz} = \partial_k \Gamma_z - \partial_z \Gamma_k + i[\Gamma_z, \Gamma_k]$ denoting the non-Abelian field strength matrix. The above expression directly shows the geometric and topological properties of the displacement generated in quantum walks and relates the properties of the walk to the Wilczek-Zee²⁵ connection along z and k defined as

$$[\Gamma_z]_{\ell m} = \langle \phi_\ell(k, z) | i\partial_z | \phi_m(k, z) \rangle \quad (16)$$

and

$$[\Gamma_k]_{\ell m} = \langle \phi_\ell(k, z) | i\partial_k | \phi_m(k, z) \rangle \quad (17)$$

Using Eq. (16) one can explicitly calculate the holonomies generated in the pumping cycles $\mathcal{C}_T$ and $\mathcal{C}_R$.

Specifically, denoting as $W_{\mathcal{C}_T}^{\xi s}$ the holonomy corresponding to the cycle $\mathcal{C}_T^{\xi s}$ on the hyperplane $J_c = sJ_d$ with $s = \pm$ with anti-clockwise ($\xi = +1$) or clockwise ($\xi = -1$) orientation shown in Fig. 2(c), we can write

$$W_{\mathcal{C}_T}^{\xi s} = e^{i\xi \frac{k}{2}(\sigma_0 - s\sigma_z)} = \begin{pmatrix} e^{i\xi \frac{k}{2}(1-s)} & 0 \\ 0 & e^{i\xi \frac{k}{2}(1+s)} \end{pmatrix} \quad (18)$$

In contrast, the holonomy associated to the cycle $\mathcal{C}_R$ defined on the plane $\{J_{b2} = J, J_{b1} = 0\}$ and shown in Fig. 2d is

$$W_{\mathcal{C}_R} = e^{i\theta(\sin k\sigma_x + \cos k\sigma_y)} = \begin{pmatrix} \cos \theta & e^{ik} \sin \theta \\ -e^{-ik} \sin \theta & \cos \theta \end{pmatrix} \quad (19)$$

where the angle θ can be expressed by the integral along the path $\mathcal{C}_R$ as

$$\theta = \int_{\mathcal{C}_R} \frac{(J_c \partial_z J_d - J_d \partial_z J_c) \sin k}{\rho^2 \Delta^2(k)} \quad (20)$$

Implementation of ThQWs in photonic waveguides and integrated photonics

The implementation of ThQWs poses several technical challenges, associated with the realization of a large number of non-Abelian holonomic transformations. So far, most experimental work has focused on implementations based on laser-written glass structures such as those realized in ref.⁴⁰ and on silicon photonics implementations⁴¹. In realistic experiments based on femtosecond laser-written glass structures, the waveguides typically have dimensions on the order of 5–10 μm , with

separations ranging from 10 to 20 μm . Typical effective refractive indices and operating wavelengths in vacuum are $n_e = 1.5$ and $\lambda_0 = 0.8 \mu\text{m}$, respectively. These values lead to an effective optical potential per waveguide of approximately $V_0 \simeq 1 \text{ meV}$ and a field decay length of a few micrometers. The corresponding coupling constants in this regime are typically on the order of $J \simeq 10^{-3} \mu\text{m}^{-1}$. To satisfy the adiabatic condition, the modulation wavelength λ must fulfill $\lambda J \gg 1$. For instance, setting $\lambda = 2 \text{ cm}$ gives $\lambda J \sim 20$, which allows approximately 10 modulation cycles within waveguides of length 20 cm. This length is realistic and permits the neglect of loss effects. Reducing the losses or finding strategies to perform non-adiabatic holonomic gates would significantly enhance the maximum number of steps. A comprehensive discussion of non-Abelian holonomic effects, derived from coupled-mode theory in photonic waveguide lattices, can be found in refs.^{47–52}.

Robustness of ThQWs

In practical implementations various imperfections and non-idealities can affect the performance of ThQWs. Deviations from the ideal pumping cycle – such as incomplete loops or distorted parameter trajectories – may, in general, reduce the fidelity of holonomic gates. However, their impact on the shift and rotation cycles is qualitatively different. The topological protection inherent in Thouless pumping ensures that such imperfections introduce only exponentially suppressed corrections to the shift cycle $\mathcal{C}_T$, as long as the energy gap protecting the degenerate subspace remains open throughout the evolution. In contrast, the rotation cycle $\mathcal{C}_R$ is only geometrically protected and is therefore more sensitive to imperfections. As a result, we expect that deviations from ideal pumping cycles lead to small step-to-step fluctuations in the rotation angle θ . The effect of this kind of dynamical disorder has been discussed e.g. in refs.^{53,54}, and it critically depends on the disorder strength. Clearly, a strong lattice disorder breaks translational symmetry and can induce localization, potentially hindering quantum walks.

Imperfect control on the Hamiltonian parameters may affect the degeneracy or the flatness of the energy bands involved in the QW. These effects introduce dynamical phase errors and wavepacket dispersion, which in turn degrades holonomic precision and coherence. To a certain extent, as discussed in refs.^{1,22,31}, these effects can be controlled by appropriately choosing the duration of the cycle in such a way that the pumping process does not resolve the small energy splitting introduced by weak control of the Hamiltonian parameters. Loss mechanisms such as photon decay or dephasing introduce non-unitarity, reduce the visibility of topological transport and essentially limit the number of steps in quantum walks,



which creates a trade-off with the adiabaticity condition. Furthermore, non-adiabatic effects arising from rapid parameter modulation lead to interband transitions and incomplete holonomies, undermining both the quantization of transport and the desired entanglement patterns. Despite these challenges, the topological nature of Thouless pumping and the geometric robustness of holonomies offer resilience to moderate imperfections. Nevertheless, achieving high-fidelity control in THQWs requires careful mitigation of non-idealities through optimized cycle design, flat-band engineering, and coherence-preserving platforms, as shown in the Table below.

Non-ideality	Effect	Model and correction strategy
Cycle fabrication inaccuracies	• Deviation from flat band condition • Imperfect or non-adiabatic cycles	Identification and optimization of robust cycles
Refractive index inhomogeneity, disorder	• Transitions between eigenstates • Reduced C_R fidelity	Accurate cycle design to reduce the impact of disorder
Imperfect preparation of the initial state	• Dispersive propagation • Apparent gate infidelity	Optimization of the initial state
Photon losses, wavelegth and inter-waveguide crosstalk	• Reduced signal intensity and coherence • Limit on ThQWs number of steps • Unintended mode mixing	• Optimal control to design non-adiabatic holonomic gates • Modeling using dissipative methods

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Data availability

The data provided in the manuscript are available upon request.

Conflict of interest

The authors declare no competing interests.

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